# More Ramanujan-type formulae for $1/\pi^2$

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One of the most spectacular achievements in the history of the number $\pi$ is the representations of $1/\pi$ by rapidly converging series discovered by S. Ramanujan [1] in 1914. Although Ramanujan himself did not indicate how he arrived at his series, he hinted that these series belong to what is now known as 'the theories of elliptic functions to alternative bases'. The first rigorous mathematical proofs of Ramanujan's identities in [1] and their generalizations were given by the Borweins [2] and the Chudnovskys [3] (see also [4] and [5]). One of the nowadays standard examples of Ramanujan-type formulae is the Chudnovskys' famous formula [3]

$$\sum_{n=0}^{\infty}\frac{(\frac16)_n(\frac12)_n(\frac56)_n}{n!^3}(545140134n+13591409)\cdot\frac{(-1)^n}{53360^{3n}}=\frac{3\cdot 53360^2}{2\pi\sqrt{10005}}, \tag{1}$$

which enabled them to hold the record in the calculation of $\pi$ in 1989–94. Here $(a)_n=\Gamma(a+n)/\Gamma(a)=a(a+1)\cdots(a+n-1)$ for $n\geqslant 1$ and $(a)_0=1$ is the Pochhammer symbol. Quite recently, following a different method J. Guillera [6], [7] managed not only to prove some of Ramanujan's identities but also to indicate similar rapidly converging series for $1/\pi^2$. In this note we present a simple algorithm for producing Ramanujan–Guillera-type formulae for $1/\pi^2$ from known ones for $1/\pi$.

The general form of a Ramanujan-type series for $1/\pi$ is as follows:

$$\alpha\cdot v(z_0)+\beta\cdot\theta v(z_0)=\frac{1}{\pi}, \qquad \theta=z\frac{\mathrm{d}}{\mathrm{d}z}, \tag{2}$$

where $\alpha$, $\beta$ and $z_0$ are certain algebraic numbers, and $v(z)$ is the analytic solution (around the origin) of a certain 'arithmetically nice' linear differential equation of order 3 normalized by $v(0)=1$. It *always* happens that the differential equation is the (symmetric) square of a differential equation of order 2, $\theta^2u+A(z)\theta u+B(z)u=0$, $A(z),B(z)\in\mathbb{Q}(z)$, say. In other words, we have $v(z)=u(z)^2$, where $u(z)$ is the analytic solution of a linear differential equation of order 2. In particular, this fact implies

$$v=u^2, \qquad \theta v=2u\,\theta u. \tag{3}$$

Consider the fourth power of the same differential equation of order 2; its analytic solution around the origin is $w(z)=u(z)^4$, that is,

$$w=u^4, \quad \theta w=4u^3\theta u, \quad \theta^2w=12u^2(\theta u)^2+4u^3\theta^2u=12u^2(\theta u)^2-4Au^3\theta u-4Bu^4. \tag{4}$$

Comparing (3) and (4) we find that

$$v^2=w, \quad v\,\theta v=\frac12\theta w, \quad (\theta v)^2=\frac43Bw+\frac13A\theta w+\frac13\theta^2w. \tag{5}$$

Taking the square of both sides of (2) and using (5) we finally arrive at the identity

$$\Big(\alpha^2+\frac43B(z_0)\beta^2\Big)\cdot w(z_0)+\Big(\alpha+\frac13A(z_0)\beta\Big)\beta\cdot\theta w(z_0)+\frac13\beta^2\cdot\theta^2w(z_0)=\frac{1}{\pi^2}, \tag{6}$$

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and the latter is a Ramanujan–Guillera-type formula for $1/\pi^2$. It is clear that taking the cube, fourth power, etc. of the both sides of (2) leads one to similar (but more complicated) formulae for $1/\pi^3$, $1/\pi^4$, … .

Our two illustrative examples are related to the identities

$$w(z)=\sum_{n=0}^{\infty}a_n\frac{(\frac13)_n(\frac23)_n}{n!^2}z^n={}_3F_2\left(\begin{matrix}\frac16,\ \frac12,\ \frac56\\ 1,\ 1\end{matrix}\,\middle|\,z\right)^2={}_2F_1\left(\begin{matrix}\frac1{12},\ \frac5{12}\\ 1\end{matrix}\,\middle|\,z\right)^4$$

involving ${}_2F_1$- and ${}_3F_2$-hypergeometric series (see, e.g., [8] for the corresponding definition), where the sequence

$$a_n=\sum_{k=0}^{n}\frac{(\frac12)_k^3}{k!^3}\,\frac{(\frac12)_{n-k}}{(n-k)!}=\sum_{k=0}^{n}\left(\frac{(\frac14)_k(\frac34)_{n-k}}{k!(n-k)!}\right)^2$$

satisfies the polynomial recurrence equation

$$8(n+1)^3a_{n+1}-(2n+1)(8n^2+8n+5)a_n+8n^3a_{n-1}=0$$

(see [9; Section 1]). The function $u(z)={}_2F_1\left(\begin{matrix}\frac1{12},\ \frac5{12}\\ 1\end{matrix}\,\middle|\,z\right)$ is the analytic solution of the differential equation

$$\theta^2u-\frac{z}{2(1-z)}\theta u-\frac{5z}{144(1-z)}u=0.$$

Writing the Chudnovskys' formula (1) in the form

$$545140134\cdot\theta v(z_0)+13591409\cdot v(z_0)=\frac{3\cdot 53360^2}{2\pi\sqrt{10005}},$$
$$\text{where}\quad z_0=-\frac{1}{53360^3}=-\frac{1}{(2^4\cdot5\cdot23\cdot29)^3},$$

we find from (6) that

$$\begin{aligned}2228833242731534 67\cdot\theta^2w(z_0)+16670750677895547\cdot\theta w(z_0)&\\ +415634396862086\cdot w(z_0)&=\frac{160080^3}{\pi^2}.\end{aligned}$$

A more modest example is based on the Ramanujan-type formula [3]

$$\sum_{n=0}^{\infty}\frac{(\frac16)_n(\frac12)_n(\frac56)_n}{n!^3}(5418n+263)\cdot\frac{(-1)^n}{80^{3n}}=5418\cdot\theta v(z_0)+263\cdot v(z_0)=\frac{640\sqrt{15}}{3\pi},\tag{7}$$

where $z_0=-1/80^3$. Taking the square of (7) we obtain

$$198144387\cdot\theta^2w(z_0)+28855107\cdot\theta w(z_0)+1400726\cdot w(z_0)=\frac{240^3}{\pi^2}.$$

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